Margin Requirements and Portfolio Optimization: A Geometric Approach

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Abstract

Using geometric illustrations, we investigate what implications of portfolio optimization in equilibrium can be generated by the simple mean-variance framework, under margin borrowing restrictions. First, we investigate the case of uniform marginability on all risky assets. It is shown that changing from unlimited borrowing to margin borrowing shifts the market portfolio to a riskier combination, accompanied by a higher risk premium and a lower price of risk. With the linear risk-return preference, more stringent margin requirements lead to a riskier market portfolio, contrary to the conventional belief. Second, we investigate the effects of differential marginability on portfolio optimization by allowing only one of the risky assets to be pledged as collateral. It is shown that the resulting optimal portfolio is not always tilted towards holding more of the marginable asset, when the margin requirement is loosened.

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*All errors are mine. First draft: July 2013.

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The latest financial crisis has rekindled interest from academics and practitioners alike in the role of margin or collateralized borrowing in the making of the meltdown. For example, the widespread and unscrupulous use of short-term repurchase agreements ("repos") is often cited as one of the culprits underlying the crisis (Gorton, 2009; Gorton and Metrick, 2012). In fact, repos are just another innovative form of margin borrowing that, due to its flexibity and discreetness, enables market players to circumvent much of the scrutiny from regulators.[1]

Financial economists have recently made substantial progress in understanding factors that may impact the margin requirements and vice versa, such as market trading liquidity (Brunnermeier and Pedersen, 2009), background liquidity risk (Wang, 2013), or volatility of asset returns (Rytchkov, 2014).[2] With the backdrop of the latest crisis, researchers have investigated, both theoretically and empirically, the impacts of the dynamics of margin requirements on market or economic outcomes, such as dislocation in equity markets (Khandani and Lo, 2007, 2011), risk-free interest rate and collaterized interest-rate spreads (Garleanu and Pedersen, 2011), low returns for high-beta assets (Frazzini and Pedersen, 2014), and the propagation of business cycles (Ashcraft *et al.*, 2011). Along with some earlier contributions (Kupiec and Sharpe, 1991; Chowdhry and Nanda, 1998), the theoretical underpinnings of this literature are dynamic competitieve equilibria with heterogeneous agents, in which the market clearing process is painstakingly modeled.

While acknowledging all of these important contributions cited above, in this paper we take one step back and asks: what can we learn from the basic mean-variance (henceforth MV) framework to understand the effects of exogenous margin changes, in particular, the effect on the riskiness of market portfolio?[3] This question is interesting because the MV optimization is still one of the canonical models used in portfolio construction and analysis by practitioners. Deriving implications from an extension of this canonical model can demonstrate its capability and limits. This will be complementary in deepening our appreciation of the more recent and sophisticated works.

Our analysis consists of two parts. First, a uniform margin rate is imposed upon all of the risky assets. This corresponds to the case of portfolio margin borrowing, that is, the entire portfolio of

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[1]Repos are short-term borrowing transactions, in which a borrower sells liquid assets to a lender with a contractual committment to buy them back at a prespecified price. The market value of collateralized assets exceeds the amount of cash the borrower receives, with the difference referred to as "haircut".

[2]In Rytchkov (2014), margin requirements are tied to market conditions, including volatility of returns, in a very general fashion, and would in turn affect these market conditions in equilibrium.

[3]Riskiness and volatility are not distinguishable in this paper due to the static nature of the MV model.

risky assets can be pledged as a whole to borrow funds. The efficient frontier of total portfolios with margin borrowing can be viewed as part of a transformed hyperbola linked back to the original hyperbola comprised only by risky assets. When the unlimited borrowing regime is replaced with margin borrowing, we show that the new market portfolio of risky assets is riskier. Furthermore, with the assumption of linear mean-variance preference, we show that more stringent margin requirements lead to riskier market portfolios.

The anlysis of uniform margin borrowing speaks to a large body of empirical literature that has examined the relationship between margin requirements and aggregate stock market.[4] This has been a very active area since the late 1980s and early 1990s, in the wake of October 1987 crash. Most researchers conclude that the margin requirements, stipulated in Regulation T by the Federal Reserve System, had little or even positive impact on stock market volatility (Ferris and Chance, 1988; Hsieh and Miller, 1990; Kumar *et al.*, 1991; Kupiec, 1989; Salinger, 1989; Schwert, 1989). The lone researcher taking the other side of the debate is Gikas Hardouvelis who, in a series of papers (Hardouvelis, 1988, 1990; Hardouvelis and Theodossiou, 2002), claims that margin requirements were indeed instrumental in reducing market volatilities, through the "pyramiding-depyramiding" process fueled by speculative investors (Garbade, 1982).[5]

Some works cited above contain the seemingly counterfactual evidence suggesting a positive relationship between margin requirements and market volatilities (Ferris and Chance, 1988; Kumar *et al.*, 1991). Several authors have proposed various theories that could predict such a positive relationship, such as coporate financing leverage response (Goldberg, 1985), heteregeneous information possessed by market participants (Ferris and Chance, 1988), market liquidity effect (Kumar *et al.*, 1991), investor heterogeneity in risk tolerance (Kupiec and Sharpe, 1991), and hetereogeneous background liquidity shocks (Wang, 2013). To complement these works, our analysis

[4] The definition of margin in stock market is the minimum percentage of equity an investor must deposit in his account to secure the loan that is used towards purchasing or maintaining his stock holidngs. When the stock value declines, the investor is forced to either post more capital or to sell some shares of stock to bring the equity share back to the specified level. The percentage of cash or securities must be deposited in the initial purchase of stock holdings is called the *initial margin*. Once an investor has bought a security on margin, the required minimum percentage of equity that must be maintained in the investor's margin account is called the *maintenance margin*. The requirements on these two margins may differ with each other. To simplify the discussion without affecting main points, we do not make this distinction in our analysis.

[5] See Fortune (2001) for an excellent summary of this debate. Hardouvelis and Peristiani (1992) also find effective evidence of margin requirements on volatilities in the stock market of Japan. Other impacts on the stock market upon changes of margin requirements are also examined, such as stock price movements (Largay and West, 1973), returns and trading volumes (Grube *et al.*, 1979), or margin credit (Luckett, 1982). Salinger (1989) and Fortune (2001) highlight the role of margin loans in affecting market volatility instead of margin requirement itself.

provides an alternative, straightforward explanation for this positive relationship based upon a classical, well-known framework.

In the second part of our analysis, we consider the case of margin borrowing only allowable for one of the two risky assets, an extreme example of differential marginability.[6] Depending upon whether the marginable security is less risky or riskier, the resulting efficient frontier of margined portfolios would be more or less concave than the efficient frontier of margined portfolios under uniform marginability. Either way, it is ambiguous with regard to whether the resulting optimal portfolio of risky assets would become riskier or not, even with the assumption of linear mean-variance preference. The ambiguity arises because, on one hand, the non-marginability of one of the assets reduces the holdings of all assets, for the effective margin borrowing capacity is reduced; on the other hand, the marginable asset becomes more valuable, thus more holdings of it are desirable, even though this means the original optimal portfolio risk-return relationship will be stretched. The optimal relative holdings of the marginable asset thus depend on the balance of these two opposing forces.

The ambiguous prediction on the demand for the marginable security is consistent with mixed evidence in studies that examine the effects of changing marginability of individual securities. Largay (1973) and Eckardt and Rogoff (1976) report the imposition of 100% margin restrictions on some stocks was associated with the termination of the upward price movement, a reduction in trading volume and a decline in volatility. Seguin (1990) finds that margin eligibility of Over-The-Counter (OTC) issues increases their post-annoucement trading prices and volumes, but not volatilities. However, Grube and Joy (1988) demonstrate that relative return variances of OTC issues declined before they were added onto the margin eligibility list administered by Fed, but not after, and no important changes in volumes are found for these issues before and after the list date. Pruitt and Tse (1996) fail to detect significant differences in price movements or volatility responses between marginable and non-marginable OTC issues after margin level changes.

Our analytical approach is primarily geometric. The properties of optimal portfolio composition with portfolio margin restrictions can be expressly illustrated by diagrams. Although

[6]Brokers can tie a security's margin rate at their discretion to its issuer's market power, liquidity, capitalization size, or balance sheet strength. For example, one of the U.S. brokerage firms on its website states that it "may reduce the collateral value of securities (reduces marginability) for a variety of reasons, including: small market capitalization or small issue size; low liquidity in the collective primary/secondary exchanges; involvement in tenders and other corporate action"(`http://www.interactivebrokers.com/en/index.php?f=margin&p=stk2`, last accessed on 12/07/2013).

geometric depiction may not be as rigorous as mathematical proof (a concern voiced in Merton (1972)), it provides a heuristic understanding of how margin borrowing can be effectively constructed and absorbed into the basic MV framework. Whenever possible, we still resort to simple mathematical proofs (in Appendix A) to guide our geometric depictions.

Portfolio Optimization with Uniform Marginability

We start off the analysis assuming a uniform margin rate for each of the available risky assets. This is equivalent to "portfolio margin borrowing", namely, the entire risky portfolio can be used as collateral to borrow funds. The purpose of this section is to show that: (1) the resulting efficient frontier with portfolio margin borrowing is a segment of the capital market line joint with a segment of a hyperbola, the latter derived from the original hyperbola depicting the efficient frontier of portfolios comprised only by risky assets; (2) for an investor whose optimal investment portfolio would have involved a lot of borrowing had his borrowing been uncapped, now his new optimal portfolio under the margin requirement would be a margin binding portfolio; (3) when the margin requirement is tightened, the investor's optimal portfolio of risky assets becomes riskier under the linear risk-return preference. These facts all have implications for the market portfolio, for the market portfolio is the sum of all investors' optimal risky portfolios.

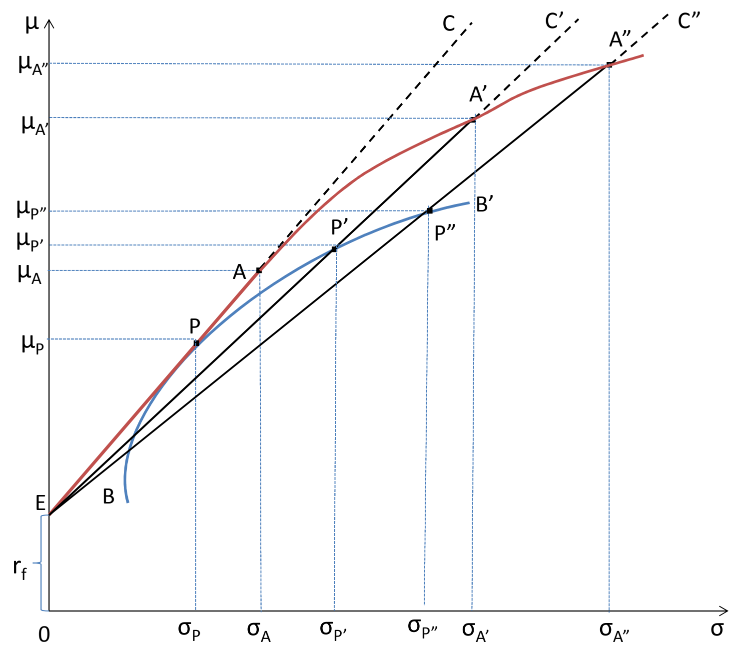
The hyperbola of efficient uniform margined portfolios

With the standard assumptions related to the standard Capital Asset Pricing Model (such as no transaction costs, infinitely divisible assets, etc.) (Elton *et al.*, 2014, Chapter13), we begin with the familiar efficient frontier attained by optimal mixes of risky assets in Figure 1 — a segment of hyperbola represented by B-B' (depicted by the solid, blue curve). This is the efficient frontier when borrowing and lending are disallowed. Now let us introduce unlimited borrowing and lending with both rates set equal to r_f.[7] With borrowing and lending possible, the efficient frontier is the straight line (the capital market line, or CML in short) tangent at the point P to the curve B-B'. This line cuts the vertical axis at the point E which is of distance r_f from the origin of the

[7] Throughout this article, the borrowing rate is assumed to be equal to the lending rate r_f. Assuming a higher borrowing rate than the lending rate does not change the main conclusions.

plot. Each point on CML can be constructed as a mixture of the optimal portfolio of risky assets, P, and a long or short position of the riskless security. P is the risky market portfolio, because every investor's total portfolio is comprised by a portion of wealth allocated to holdings of this portfolio, plus the rest allocated to a long or short position of the riskless security.

Figure 1: Constructing the portfolio efficient frontier with margin borrowing: uniform margin rates across all risky assets



Accordingly, the expected rate of return of any total portfolio A on the CML, μ_A, can be expressed as a weighted average of the expected rate of return of P, μ_P, and the riskless rate of return, r_f,

$$\mu_A = r_f + \frac{\sigma_A}{\sigma_P}(\mu_P - r_f) = \frac{\sigma_A}{\sigma_P}\mu_P + \left(1 - \frac{\sigma_A}{\sigma_P}\right) r_f , \tag{0.1}$$

where σ_A and σ_P are the return standard deviations of A and P, respectively. $\omega_P \equiv \frac{\sigma_A}{\sigma_P}$ is the weight of the value of holdings in P relative to the total portoflio net value and can be greater than one. When $\omega_P > 1$, A is located to the east of P on the CML, and the investor is borrowing funds (short the riskless security) to purchase more P, with $\omega_P - 1$ being the ratio of the borrowed amount relative to the net value of the total portfolio.

Suppose now the uniform margin restriction is imposed. Assume the margin rate is α $(0 < \alpha < 1)$. This implies that the value of the equity an investor puts into purchasing the portfolio P should be no less than α times the value of holdings of the portfolio P. Or, equivalently, the margin loan this investor borrows in order to purchase the portfolio P should be no more than $(1-\alpha)$ times the value of holdings of portfolio P,

$$\omega_P - 1 \leq (1-\alpha)\cdot\omega_P\,,$$

or,

$$0 \leq \omega_P \equiv \frac{\sigma_A}{\sigma_P} \leq \frac{1}{\alpha}\,. \tag{0.2}$$

Let A stand for the boundary portfolio the investor can hold with maximum borrowing under the margin constraint, the second inequality in (0.2) becomes an equality,

$$\frac{\sigma_A}{\sigma_P} = \frac{1}{\alpha}\,,\quad \text{or,}\quad \sigma_A = \frac{1}{\alpha}\sigma_P\,. \tag{0.3}$$

Corresponding to the plot in Figure 1, this says that A is the point on the CML such that its horizontal distance from zero is $\frac{1}{\alpha}$ times that of point P. Since the length of EA relative to that of EP on the CML is also equal to $\frac{\sigma_A}{\sigma_P}$, we have

$$\frac{EA}{EP} = \frac{\sigma_A}{\sigma_P} = \frac{1}{\alpha}\,. \tag{0.4}$$

It is important to realize that (0.4) embodies the general process of locating the boundary margined portfolios, in the presence of the margin requirement, for any combination of risky assets an investor is willing to hold, not just for the portfolio P. Any ray that originates from the point E and lies below the CML represents total portfolios comprised by a weighted mixture of a particular combination of risky assets and the riskless asset. For example, the straight line E-P'-A'-C' captures all mixtures of the risky portfolio P' (which is where the ray intersects with the risky portfolio efficient frontier B-B') and the riskless security. Due to the margin requirement and following the same reasoning leading to (0.4), the investor can borrow up to the limit of margin constraint and ends up at A', where

$$\frac{EA'}{EP'} = \frac{\sigma_{A'}}{\sigma_{P'}} = \frac{1}{\alpha}\,.$$

If we keep swinging the ray originating from E clockwise, and collect the boundary points such as A and A' and connect them one by one, eventually we obtain the efficient frontier comprised by fully margined portfolios as is represented by the curve E-P-A-A'-A''. For any portfolio located on the segment A-A'-A'', the margin constraint is binding. Since the curve A-A'-A'' is constructed from the hyperbola segment P-P'-B', A-A'-A'' is also a segment of a hyperbola.[8]

Optimal portfolio choice under uniform marginability

With the imposition of the uniform margin requirement, the points of total portfolio northeast to A — the line segment A-C — become infeasible to investors and are drawn as a black, dotted line. An investor who would have chosen an optimal portfolio on the segment A-C — would have borrowed beyond what the margin restriction permits — now has to select one of feasible portfolios beneath the CML, and more specifically, on the margin efficient frontier A-A'-A''. That is, the investor ends up choosing a margin binding portfolio.

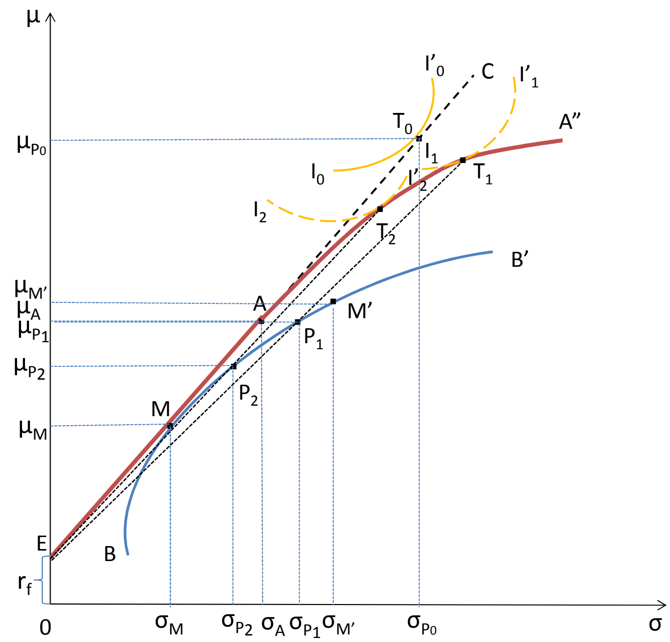
In Figure 2, we sketch the transition an investor is forced to make when the borrowing opportunity set is shifted from being unlimited to being subject to the margin constraint. Assume an investor's previous optimal portfolio with borrowing was T_0 on the segment A-C in Figure 2. T_0 is determined by the tangency of the CML to one of the investor's indifference curves I_0-I_0' (the yellow, solid curve).[9] Now, when A-C is no longer available and the margin requirement is imposed, the red, solid curve A-A'' is the efficient frontier on which the investor is able to choose an optimal portfolio. Depending on the exact preference profile of this investor, he might find the portfolio T_1 being the optimal choice (tangent with the indifference curve I_1-I_1'), or the portfolio T_2 (tangent with the indifference curve I_2-I_2'). Notice that, from the diagram, T_1 is riskier than T_0, and T_2 is less riskier. Yet, both T_1 and T_2 are margin binding portfolios, for every portfolio on the curve A-A'' is margin binding.

When the investor's optimal total portfolio is shifted from T_0 to T_1 or T_2, his optimal combination of risky assets has changed. For example, when we connect the point T_1 with E through a straight line, it crosses the hyperbola segment B-B' at the point P_1. P_1 is thus the risky asset combination within the total portfolio T_1. In other words, T_1 is a mixture of a long position in the risky portfolio P_1 and a short position in the riskless security. Likewise, the risky asset combination

[8] Appendix A.1 contains its mathematical proof.

[9] Portfolios on the indifference curve I_0-I_0', although with varying risk-return combinations, yield the same level of utility or satisfaction to the investor. Different indifference curves correspond to different levels of utility or satisfaction. For a particular investor, none of his indifference curves should cross any other.

Figure 2: Investor's optimal portfolio choice: from unlimited borrowing to portfolio margin borrowing



within the total portfolio T_2 can be located at the point P_2 in a similar manner. Suppose M was the market portfolio of risky assets with unlimited borrowing. Notice that, even though the new total portfolio T_1 or T_2 can be riskier or less risky than the original T_0, their underlying risky asset combinations P_1 and P_2 are both located to the right of the origional risky market portfolio M and thus are both riskier than M.

To summarize, when investors shift their optimal total portfolios away from the line segment A-C to the curve A-A'' due to the change of unlimited borrowing regime into margin borrowing regime, their current risky asset combinations are always riskier and are located to the right of M, regardless of whether their current total portfolios are riskier or not than before. Consequently, current market portfolio of risky assets, which is the sum of risky asset combinations held by all investors, must be riskier than M and must be located to the right of M, such as the point M' in Figure 2.

The location of M' is also above M. Therefore, $\mu_{M'} > \mu_M$. The risk premium, measured by $\mu_{M'} - r_f$, now is higher than the previous value, $\mu_M - r_f$. The current market price of risk, measured by $\frac{\mu_{M'} - r_f}{\sigma_{M'}}$, is the slope of the straight line connecting E with M'. Since this slope is lower than that of the straight line connecting E with M, the current price of market risk is lower than the previous one when M was the risky market portfolio.

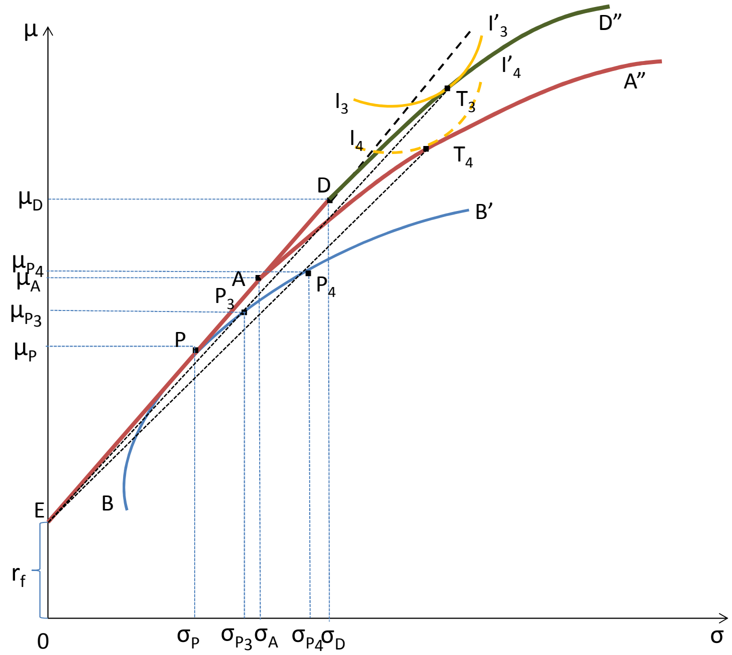
Optimal margined portfolios when uniform margin rate is changed

Following the same analysis, it turns out raising margin requirements does not necessarily lead to a less risky market portfolio. In Figure 3, we depict the efficient frontiers corresponding to two different margin requirements. The curve P-A-A'' is the efficient frontier carried over from Figure 1, and α is the margin rate associated with it. Suppose the margin requirement is α', where $\alpha' < \alpha$. Since investors can borrow more with the same amount of equity, the hyperbola curve starts somewhere further along the CML E-P-A, say, at the point D. This is so because, to borrow at the maximum capacity, we have

$$\frac{ED}{EP} = \frac{\sigma_D}{\sigma_P} = \frac{1}{\alpha'}. \tag{0.5}$$

$\alpha' < \alpha$ implies D is northeast to A along the CML. The part of the hyperbola curve D-D'' starts with the point D and is derived in the same fashion as is A-A''.

Figure 3: Investor's optimal portfolio choice under margin restrictions: varying uniform margin rates



Suppose the optimal portfolio for an investor at the margin rate α' is T_3, which is the tangent point of the indifference curve I_3-I'_3 to the curve D-D''. Now consider the margin rate being changed from α' to α, corresponding to the tightening of margin borrowing. The new optimal portfolio, at the margin rate α, is the tangent point of the new indifference curve I_4-I'_4 (parrallel to but located south to I_3-I'_3) to the curve A-A'', denoted by T_4. Without further details of the investor's risk-return preference, it is difficult to conclude whether the portfolio T_4 is less or more risky than T_3.

If an investor possesses the linear risk-return preference, Appendix A.2 proves that the new optimal portfolio T_4 has a lower expected return and a lower return standard deviation than does T_3, that is, T_4 is located to the southwest of T_3. With that, tracing out the underlying risky asset combinations of T_3 (denoted by P_3) and of T_4 (denoted by P_4) follows the same procedure as is tracing out P_1 and P_2 in Figure 2. That is, P_3 (or P_4) is the point located by connecting T_3 (or T_4) with E and crossing the curve B-B'. Since T_4 is now to the southwest of T_3, the relative positions of P_3 and P_4 to each other appear ambiguous: P_4 might be to the northeast of P_3, thus is riskier than P_3, or to the southwest of P_3, thus is less risky than P_3. Again, we prove in Appendix A.2 that, given the linear risk-return preference, P_4 is unambiguously located to the northeast of P_3, thus is riskier than P_3.

All of those optimal portfolios previously located on the curve D-D'', such as T_3, now are forced to move to the curve A-A'' due to the increase of margin rates, leading to the shift of underlying risky asset combinations to riskier positions. Furthermore, all of those investors whose optimal portfolios were on the line segment A-D were holding the underlying risky asset combination P, but are now choosing new ones on the curve A-A'', of which every point corresponds to a riskier combination of risky assets than P on the curve P-B'. Therefore, the market portfolio of risky assets is riskier when the margin requirement is tightened, thanks to the assumption of linear risk-return preference. This in turn leads to a higher risk premium and a lower price of risk.

It is interesting to obtain this essential result of margin borrowing within the classical MV framework, just with one additional assumption of linear mean-variance preferences for investors. The risk preferences of investors are fixed but are not necessarily homogenous. The tightening of margin requirements effectively pushes the market portfolio towards a riskier position. Outside the MV framework, we are certainly not the only one suggesting a positive relationship between margin requirements and market volatility. An earlier paper by Goldberg (1985) argues that in the presence of margin restrictions on investors, firms would act in the best interests of their shareholders by leveraging up to offset the margin restrictions. This would increase the stock price volatility. Ferris and Chance (1988) postulate that margin reductions permit more investors to enter the market, bringing in with them more heterogeneous information. Therefore, investors are less likely to engage in unidirectional transactions that may contribute to higher volatilities. In both Kupiec and Sharpe (1991) and Wang (2013), investors exhibit heterogeneous risk preferences, and risk-bearing outcomes in the economy may increase or decrease stock market volatility. For instance, in Wang (2013), if liqudity suppliers are more constrained by margin requirements than are liqudity demanders, then market volatility is increased.

On the other hand, our results are in contrast to those in Rytchkov (2014), who shows that when the margin requirement are contingenet on market conditions, such as the volatility of returns, in a very general way, and when the margin constraint is binding, in equilibrium the corresponding risk-free rate is lower, the volatility of return is lower, and the market price of risk and risk premium are both higher. In our model, the change of margin requirements is not contingent on any of the market conditions. It is more appropriate to interpret our results as the impacts on market portfolios when there is an exogenous shock on margin requirements. This is indeed, implicitly or otherwise, assumed in most of the empirical literature.

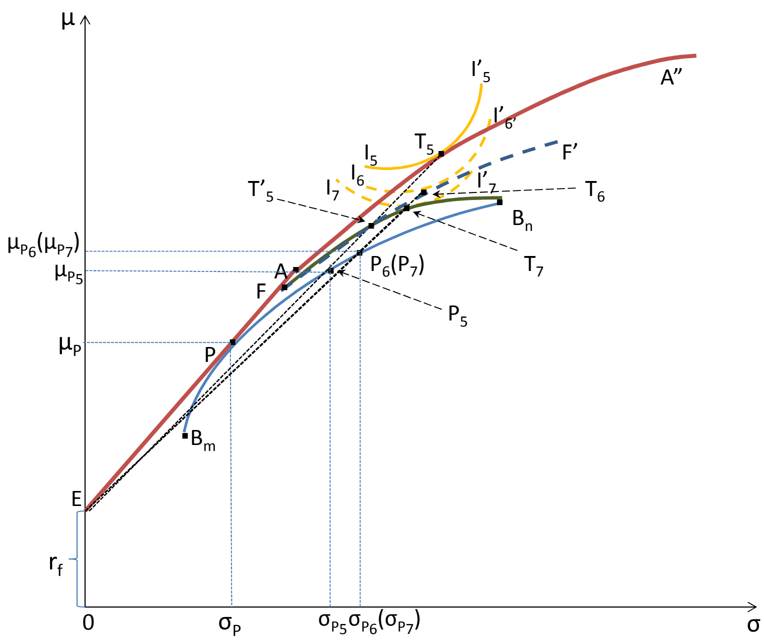
Portfolio Optimization with Differential Marginability

In this section, we consider the case of differential margin rates applied to different risky assets. To simplify the exposition, we assume that only two risky assets are available, and short-sales are disallowed. To bring out the sharp contrast into focus, let us assume that only one of the two risky assets can be used as collateral for borrowing. This marginable aset can be the less risky of the two, or the riskier of the two. In the analysis that follows, we will often switch back to the case of uniform margin requirements for comparison. The unambiguous conclusion is that there are no unambiguous implications on portfolio optimization, thus no unambiguous impacts on risky market portfolio, even when the linear mean-variance preference is assumed. This can be observed from the complex hyperbola equation (A.19) in Appendix A.3, from which no general insights can be obtained unless further restrictions on parameter values are imposed. The result of no unambiguous results is useful to know, for it counters the conventional believe that, other things equal, a risky asset that becomes more marginable would be in higher demand by investors.

The less risky asset is marginable

Assume the marginable asset m has a lower risk and a lower rate of return than does the non-marginable asset n. In Figure 4, the point B_m stands for the risk-return profile of the marginable asset, and the point B_n, of the non-marginable asset. The marginability of the asset m implies that its associated margin rate is still α, whereas the non-marginability of the asset n implies that its associated margin rate is in fact 1. The hyperbola curve B_m-B_n represents all of the efficient portfolios with positive weights in B_m and B_n. Again, with the lending rate r_f, the CML is tangent to the curve B_m-B_n at the point P.

Figure 4: Investor's optimal portfolio choice under margin restrictions: only the less risky asset is marginable



Let us re-examine the underlying risky portfolio P. In previous case of uniform margin borrowing at the rate α, the point A was the binding margined portfolio that can be attained by holding the risky portfolio P. Recall that the segment of hyperbola curve A-A'' is the efficient frontier of total portfolios when the uniform margin constraint is binding over the entire range of B_m-B_n. Now, instead of the whole portfolio P, only the holdings of asset B_m in the portfolio P can be used for margin borrowing. This effectively lowers the maximum borrowing capacity of the portfolio P. Rather than the point A along the CML, now the point F, which is closer to the origin than A, is the binding margined portfolio with P. So, the restriction that only the asset m can be used for margin borrowing (at the same margin rate α) leads to

$$\frac{PF}{EP} < \frac{PA}{EP}\,. \tag{0.6}$$

Suppose the asset share of m in the portfolio P is ω_P, then,

$$\frac{EP}{EF} = \omega_P \times \alpha + (1-\omega_P) \times 1 = \alpha + (1-\omega_P)(1-\alpha) > \alpha = \frac{EP}{EA}\,, \tag{0.7}$$

or equivalently,

$$\frac{PF}{EP} = \frac{\omega_P(1-\alpha)}{\omega_P\alpha + 1 - \omega_P} < \frac{PA}{EP} = \frac{1}{\alpha} - 1\,. \tag{0.8}$$

The next step is to trace out other points on the efficient frontier of margined portfolios while maintaining the assumption that only the asset m is marginable. It is done by swinging the ray originating from E clockwise, starting with the CML position. The intersection point of this ray with the curve B_m-B_n is the risky portfolio used as collateral for borrowing. For example, the ray E-P_5 crosses the curve B_m-B_n at P_5, and P_5 is the underlying risky portfolio. Should both assets be marginable, the cross point on the curve A-A'' would be T_5, where T_5 would be the fully margined portfolio with

$$\frac{P_5T_5}{EP_5} = \frac{1}{\alpha} - 1\,. \tag{0.9}$$

But now only m is marginable. Compared with P, the risky portfolio P_5 includes a smaller share of the asset m, for P_5 is farther away from the point B_m than is P. Therefore, now the amount of loans that can be secured by the same value of holdings in P_5 is less than that by the same value of holdings in P. Extend E-P_5 to T'_5, where P_5-T'_5 represents the proportion of maximum borrowing that can be attained based upon the asset composition of P_5. Denoting the share of the marginable asset m in the portfolio P_5 by $\omega_{P,5}$, we obtain a similar equation to (0.8),

$$\frac{P_5T'_5}{EP_5} = \frac{\omega_{P_5}(1-\alpha)}{\omega_{P_5}\alpha + 1 - \omega_{P_5}}\,, \tag{0.10}$$

which can be used to pin down the exact metrics of T'_5. The fact that $\omega_{P_5} < \omega_P$, along with the equations (0.8) and (0.10), indicates

$$\frac{P_5T'_5}{EP_5} < \frac{PF}{EP}, \tag{0.11}$$

that is, along the efficient frontier of the risky portfolio in the direction from B_m to B_n, the fully margined portfolio that can be supported by the corresponding risky portfolio on the curve B_m-B_n gets closer to the risky portfolio itself in terms of the distance between these two portfolios on the ray originating from E. At the point B_n, the risky portfolio consists only of the non-marginable asset n itself, and the margined portfolio concides with the point B_n perfectly.

Connect all of the margined portfolios traced out this way and we obtain the curve F-B_n in Figure 4. The marginability of only the asset m effectively reduces the overall marginability of risky portfolios on P-B_n, and more so from P to B_n, for the share in m is decreasing from P to B_n. As the result, F-B_n is more concave than the curve A-A'' in its shape.

Based upon (0.7), define the effective portfolio margin rate α as a function of ω,

$$\alpha(\omega) \equiv \alpha + (1-\omega)(1-\alpha), \tag{0.12}$$

where ω is the value of portfolio share in the asset m. For example, at the point F, as we have shown, the effective portfolio margin rate is $\alpha(\omega_P)$. Let us conterfactually assume both assets are marginable, and the portfolio margin rate is at $\alpha(\omega_P)$ throughout, to trace out the corresponding efficient frontier F-F' (the dashed, blue curve). So F-F' is essentially the efficient frontier for the case of portfolio margin borrowing when margin is set at $\alpha(\omega_P)$ for all of the risky portfolios from P to B_n.

Assume the same linear risk-return preference for the investor. One of his indifference curves I_5-I'_5 is tangent to the curve A-A'' at T_5, and another, I_6-I'_6, tangent to the curve F-F' at T_6. Even though it is not so obvious from Figure 4, but by the proof in Appendix A.1, T_6 is to the southwest of T_5 with a steeper risk-return substitution rate, and the underlying risky portfolio P_6 of T_6 lies to the northeast of P_5 of T_5.

Enter the efficient frontier of differentially margined portfolios F-B_n. With ω being decreased all the way from F to B_n and thus $\alpha(\omega)$ is increased, it is easy to see that the tangent line to every point on F-B_n is flatter than that on F-F', for the same level of risk on the horizontal axis. This implies the tangent point of the investor's indifference curve to the curve F-B_n, denoted by T_7, must lies southwest to T_6 in order to stretch for a steeper risk-return tradeoff. Since T_6 is already to the southwest of T_5, T_7 is as well.

Exactly to what degree that T_7 is to the southwest of T_5 is unknown, though. That is determined by all the parameters packed into the equation (A.19). Therefore, we cannot determine whether the underlying risky portfolio of T_7, P_7, is or is not to the northeast of the underlying risky portfolio of T_5, P_5, like what we have proved for the case of uniform margin rates. In Figure 4, we draw a P_7 that is fairly close to P_6 and is indeed to the northeast of P_5, but this is in no way guaranteed. In other words, we are unable to show unambiguously whether only keeping the marginability of the asset m increases or decreases its demand from the investor.

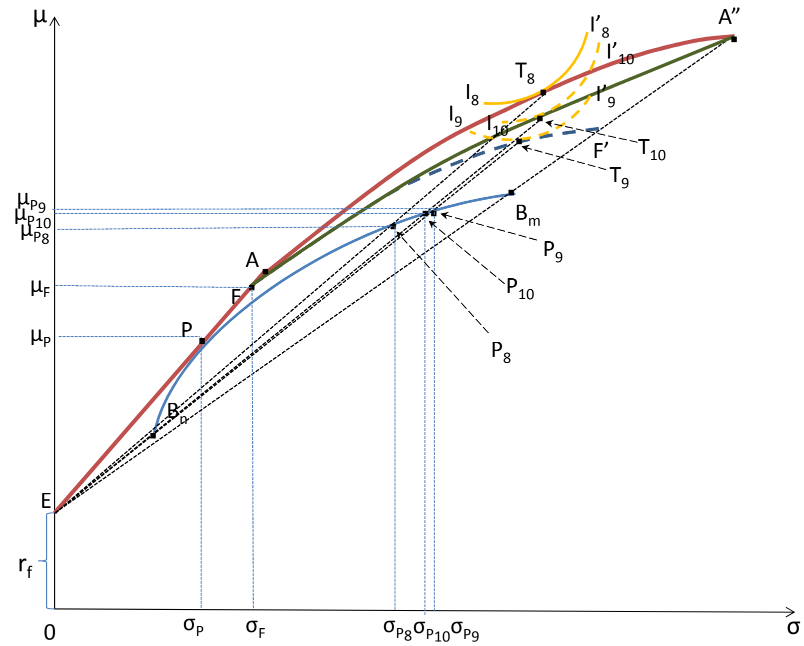
The more risky asset is marginable

Switching the risk-return profiles between marginable and non-marginable assets will not change these conclusions. In Figure 5, the marginable asset m has both a higher rate of return and a higher return standard deviation than does the non-marginable asset n, in contrast to Figure 4. B_m is now the most northeast point of the efficient frontier of risky portfolios B_n-B_m. At B_m, the portfolio is only comprised by m, and the fully margined portfolio is at the point A'', coinciding with the end point of efficient frontier of uniform margined portfolios (recall that P-A-A'' is the efficient frontier of fully margined portfolios under uniform marginability at the margin rate α).

With only the asset m marginable, the (green, solid) curve F-A'' depicts the corresponding efficient frontier of margined portfolios. Moving from P to B_m on the curve P-B_m corresponds to a greater share of asset m in the risky portfolio, which in turn leads to a lower equivalent portfolio margin rate and a higher borrowing capacity, as is exemplified by the equation (0.12).

The leftmost end point F of the curve F-A'' is below A of the curve A-A''. F-A'' is less concave than A-B_m, due to the fact that the effective portfolio margin rate is decreasing from P to B_m. Once again, we can derive the hypothetical uniform margined portfolio efficient frontier F-F', assuming both assets are subject to the same margin rate as that at the point F (both A-A'' and F-F' are efficient frontiers derived from assuming uniform margin rates, and the difference is only that the margin rate for A-A'' is less than the margin rate for F-F'). Suppose the investor's optimal portfolio on

Figure 5: Investor's optimal portfolio choice under margin restrictions: only the riskier asset is marginable



A-A'' is T_8 (the tangency point of the indifference curve I_8-I_8' to A-A'') and that on F-F' is T_9 (the tangency point of the indifference curve I_9-I_9' to F-F'). Following previous analysis, T_9 would be located to the southwest of T_8.

At each risk level, the tangency line to the point on F-A'' is steeper than the counterpart one on A-A'', or the one on F-F'. This implies the optimal portfolio on F-A'' should have a flatter risk-return tradeoff than the one on A-A'', or on F-F'. However, this implication reveals nothing regarding whether T_{10}, the tangency point of the indifference curve I_{10}-$I_{10'}$ to F-A'', lies to the southwest of T_8, or to the southeast of T_8. Let P_8 and P_{10} represent the corresponding optimal risky portfolios in T_8 and T_{10}. Apparently, T_{10} to the southeast of T_8 clearly indicates that P_{10} is located to the right of P_8. However, T_{10} to the southwest of T_8 does not reveal any information of the relative position of P_{10} to P_8. To be precise, only when T_{10} is located to the left of the intersection of the line E-T_8 with F-A'' is it for certain that P_{10} is to the left of P_8. In Figure 5, T_{10} is drawn in such a way that P_{10} is to the right of P_8, but this is not generalizable.

Although not backed up by mathematical proofs, a tentative explanation can be offered to gain an intuitive understanding of why stripping off marginability of an asset does not necessarily spur the demand for the other asset that remains marginable. On one hand, taking away the marginability of an asset reduces the borrowing capacity of the whole portfolio of both assets, and would reduce the holdings of both assets (from T_8 to T_9); on the other hand, the relative value of the marginable asset is increased because of its remaining marginability, thus more holdings of it are expected (from T_9 to T_10), even though this may imply the original optimal risk-return tradeoff relationship of the risky portfolio will be stretched. The net balance of these two opposite forces determines whether the demand of the marginable asset is greater or less than before.

Conclusion

In this paper, we analyze a simple MV portfolio optimization model by augmenting it with a margin borrowing constraint. We consider both uniform marginability and differential marginability for risky assets included in a portfolio. In the case of uniform marginability, raising the margin requirement (imposing a higher margin rate) would result in a riskier market portfolio in the new equilibrium, when the linear mean-variance preference is assumed. Unfortunately, in the case of differential marginability, no unambiguous conclusions can be achieved. Our analysis complements existing works on margin requirements by providing an alternative, simple set of predictions based upon the canonical MV framework. Since the MV framework forms the basis of the CAMP model, the results derived in this paper could be used as a benchmark to appreciate more recent progress made in this field.

References

Ashcraft, A., Grleanu, N. and Pedersen, L.H. (2011) Two monetary tools: Interest rates and haircuts. *NBER Macroeconomics Annual* 25(1): 143–180.

Brunnermeier, M.K. and Pedersen, L.H. (2009) Market liquidity and funding liquidity. *Review of Financial studies* 22(6): 2201–2238.

Campbell, J.Y. and Viceira, L.M. (2002) *Strategic Asset Allocation: Portfolio Choice for Long-term Investors*. New York, NY: Clarendon Press.

Chowdhry, B. and Nanda, V. (1998) Leverage and market stability: the role of margin rules and price limits. *The Journal of Business* 71(2): 179–210.

Eckardt, W.L.J. and Rogoff, D.L. (1976) 100% margins revisited. *The Journal of Finance* 31(3): 995–1000.

Elton, E.J., Gruber, M.J., Brown, S.J. and Goetzmann, W.N. (2014) *Modern Portfolio Theory and Investment Analysis*. Hoboken, NJ: John Wiley & Sons.

Ferris, S.P. and Chance, D.M. (1988) Margin requirements and stock market volatility. *Economics Letters* 28(3): 251–254.

Fortune, P. (2001) Margin lending and stock market volatility. *New England Economic Review* 4: 3–26.

Frazzini, A. and Pedersen, L.H. (2014) Betting against beta. *Journal of Financial Economics* 111(1): 1 – 25.

Garbade, K.D. (1982) Federal reserve margin requirements: A regulatory initiative to inhibit speculative bubbles. In: P. Wachtel (ed.) *Crises in the Economic and Financial Structure*, Lexington, MA: Lexington Books.

Garleanu, N. and Pedersen, L.H. (2011) Margin-based asset pricing and deviations from the law of one price. *Review of Financial Studies* 24(6): 1980–2022.

Goldberg, M.A. (1985) The relevance of margin regulations: Note. *Journal of Money, Credit and Banking* 17(4): 521–527.

Gorton, G. (2009) The subprime panic. *European Financial Management* 15(1): 10–46.

Gorton, G. and Metrick, A. (2012) Securitized banking and the run on repo. *Journal of Financial Economics* 104(3): 425 – 451.

Grube, R.C. and Joy, M. (1988) Some evidence on the efficacy of security credit regulation in the otc equity market. *Journal of Financial Research* 11(2): 137–42.

Grube, R.C., Joy, O.M. and Panton, D.B. (1979) Market responses to federal reserve changes in the initial margin requirement. *The Journal of Finance* 34(3): 659–674.

Hardouvelis, G.A. (1988) Margin requirements and stock market volatility. *Federal Reserve Bank of New York Quarterly Review* (Sum): 80–89.

Hardouvelis, G.A. (1990) Margin requirements, volatility, and the transitory component of stock prices. *The American Economic Review* 80(4): 736–762.

Hardouvelis, G.A. and Peristiani, S. (1992) Margin requirements, speculative trading, and stock price fluctuations: The case of Japan. *The Quarterly Journal of Economics* 107(4): 1333–1370.

Hardouvelis, G.A. and Theodossiou, P. (2002) The asymmetric relation between initial margin requirements and stock market volatility across bull and bear markets. *Review of Financial Studies* 15(5): 1525–1559.

Hsieh, D.A. and Miller, M.H. (1990) Margin regulation and stock market volatility. *The Journal of Finance* 45(1): 3–29.

Kaplan, P.D. (ed.) (2011) *Frontiers of Modern Asset Allocation*. Hoboken, NJ: Wiley Finance.

Khandani, A.E. and Lo, A.W. (2007) What happened to the quants in august 2007? *Journal of Investment Management* 5(4): 5–54.

Khandani, A.E. and Lo, A.W. (2011) What happened to the quants in august 2007? evidence from factors and transactions data. *Journal of Financial Markets* 14(1): 1–46.

Kumar, R., Ferris, S.P. and Chance, D.M. (1991) The differential impact of federal reserve margin requirements on stock return volatility. *The Financial Review* 26(3): 343–366.

Kupiec, P.H. (1989) Initial margin requirements and stock returns volatility: Another look. *Journal of Financial Services Research* 3(2-3): 287–301.

Kupiec, P.H. and Sharpe, S.A. (1991) Animal spirits, margin requirements, and stock price volatility. *The Journal of Finance* 46(2): 717–731.

Largay, J.A.I. (1973) 100% margins: Combating speculation in individual security issues. *The Journal of Finance* 28(4): 973–986.

Largay, J.A.I. and West, R.R. (1973) Margin changes and stock price behavior. *Journal of Political Economy* 81(2): 328–339.

Lee, C.F., Finnerty, J.E. and Chen, H.Y. (2010) Risk-aversion, capital asset allocation, and markowitz portfolio-selection model. In: C.F. Lee, A.C. Lee and J. Lee (eds.) *Handbook of Quantitative Finance and Risk Management*, New York, NY: Springer US, pp. 69–92.

Luckett, D.G. (1982) On the effectiveness of the federal reserve's margin requirement. *The Journal of Finance* 37(3): 783–795.

Merton, R.C. (1972) An analytic derivation of the efficient portfolio frontier. *Journal of Financial and Quantitative Analysis* 7(4): 1851–1872.

Pruitt, S.W. and Tse, K. (1996) The price, volatility, volume, and liquidity effects of changes in federal reserve margin requirements on both marginable and nonmarginable otc stocks. In: *The Industrial Organization and Regulation of the Securities Industry*, Chicago, IL: University of Chicago Press, pp. 317–358.

Rytchkov, O. (2014) Asset pricing with dynamic margin constraints. *The Journal of Finance* 69(1): 405–452.

Salinger, M.A. (1989) Stock market margin requirements and volatility: Implications for regulation of stock index futures. *Journal of Financial Services Research* 3(2): 121–138.

Schwert, G. (1989) Margin requirements and stock volatility. *Journal of Financial Services Research* 3(2-3): 153–164.

Seguin, P.J. (1990) Stock volatility and margin trading. *Journal of Monetary Economics* 26(1): 101–121.

Wang, Y. (2013) Why can margin requirements increase volatility and benefit margin constrained investors? Technical report, Robert H. Smith School of Business, University of Maryland.

A Supplemental Material

A.1 The equations of hyperbola under uniform marginability

Define $(\widetilde{\sigma}_P, \widetilde{\mu}_P)$ as the expected return and return standard deviation of an efficient portfolio $\widetilde{P}$ comprised only by risky assets. Merton (1972) has shown that the resulting efficient frontier is the upper half of the hyperbola defined by

$$\frac{\widetilde{\sigma}_P^2}{\overline{\sigma}^2} - \frac{(\widetilde{\mu}_P - \overline{\mu})^2}{\frac{D}{C} \cdot \overline{\sigma}^2} = 1 \,, \quad \widetilde{\sigma}_P \geq \overline{\sigma} \,, \widetilde{\mu}_P \geq \overline{\mu} \,, \tag{A.1}$$

where $C > 0$, $D > 0$ (see their detailed definitions in Merton (1972)). For this hyperbola, the vertex point (where the curve makes its sharpest turn) is $(\overline{\sigma}, \overline{\mu})$, and the asymptote (a straight line to which the hyperbola converges if continued indefinitely) is

$$\widetilde{\mu}_P = \overline{\mu} + \sqrt{\frac{D}{C}} \widetilde{\sigma}_P \,. \tag{A.2}$$

The efficient frontier is comprised by points northeast to $(\overline{\sigma}, \overline{\mu})$ on this hyperbola.

The above hyperbola can be viewed as the case corresponding to the margin requirement set at one ($\alpha = 1$). Now, let $\alpha < 1$ and (σ_P, μ_P) be the risk and expected return of a portfolio P on the efficient frontier associated with margin requirements $\alpha < 1$. The relationships between μ_P and $\tilde{\mu}_P$ and that between σ_P and $\tilde{\sigma}_P$ are, respectively:

$$\frac{\widetilde{\mu}_P - r_f}{\mu_P - r_f} = \alpha \,, \tag{A.3a}$$

$$\frac{\widetilde{\sigma}_P}{\sigma_P} = \alpha \,. \tag{A.3b}$$

Solving out $\widetilde{\mu}_P$ in terms of μ_P, and $\widetilde{\sigma}_P$ in terms of σ_P and plugging them back into (A.1) yield the part of the hyperbola constituted by efficient portoflios P:

$$\frac{\sigma_P^2}{\overline{\sigma}^2/\alpha^2} - \frac{\left(\mu_P - \frac{1}{\alpha}\overline{\mu} + \left(\frac{1}{\alpha} - 1\right) r_f\right)^2}{\frac{D}{C} \cdot \overline{\sigma}^2/\alpha^2} = 1 \,. \tag{A.4}$$

The range of the pair (μ_P, σ_P) on the hyperbola curve is

$$\mu_P \geq \frac{1}{\alpha}\mu_P^l + \left(1 - \frac{1}{\alpha}\right) r_f \,, \quad \sigma_P \geq \frac{1}{\alpha}\sigma_P^l \,, \tag{A.5}$$

where (μ_P^l, σ_P^l) is the expected return and return standard deviation of the optimal portfolio of riskless and risky assets under the lending rate r_f, denoted by P^l. The right-side values in the two inequalities in (A.5) are the expected return and risk that can be attained if the optimal portfolio P^l is leveraged to the maximum under the margin rate α.

From (A.4), the first-order condition of μ_P with respect to σ_P is

$$\frac{\partial \mu_P}{\partial \sigma_P} = \frac{D}{C} \cdot \frac{\sigma_P}{\mu_P - \frac{1}{\alpha}\overline{\mu} - \left(1 - \frac{1}{\alpha}\right) r_f} > 0 \,, \tag{A.6}$$

where it is positive, because, as long as the first inequality in (A.5) holds, along with $\mu_P^l > \overline{\mu}$, $\mu_P - \frac{1}{\alpha}\overline{\mu} - \left(1 - \frac{1}{\alpha}\right) r_f > 0$ follows.

The derivative of $\frac{\partial \mu_P}{\partial \sigma_P}$ with respect to α is

$$\frac{\partial^2 \mu_P}{\partial \sigma_P \partial \alpha} = \frac{D\sigma_P}{C} \cdot \frac{\overline{\mu} - r_f}{\left((\mu_P - r_f) - \frac{1}{\alpha}(\overline{\mu} - r_f)\right)^2} \left(-\frac{1}{\alpha^2}\right) . \tag{A.7}$$

When riskless lending is available but riskless borrowing is limited, Elton *et al.* (2014, Chapter 12) has shown that the riskless lending rate should be less than the level of $\overline{\mu}$ (i.e., $\overline{\mu} > r_f$). Thus (A.7) is negative, which implies that at the same level of σ_P on the hyperbolas, as α increases — the borrowing capacity is tightening and the hyperbola curve shifts downwards — the tangent line becomes flatter.

A.2 The investor's equilibrium under uniform marginability

Assume an investor has the linear mean-variance preference defined as

$$U(\mu_P, \sigma_P) = \mu_P - \frac{\gamma}{2}\sigma_P^2 \,, \quad \gamma > 0 \,. \tag{A.8}$$

For this investor not to feel worse off, a higher rate of expected return has to be provided to compensate for a higher risk as is measured by the standard deviation of returns. Holding the investor's utility level constant at $U(\cdot) = u_0$, the optimal rate of subtitution between expected return and return standard deviation is

$$\frac{d\,\mu_P}{d\,\sigma_P}\,|_{u_0} = \gamma\sigma_P\,. \tag{A.9}$$

In equilibrium, the optimal substitution rate is set to equal the marginal return from the increase of one unit of return standard deviation, that is, (A.6) is set to equal to (A.9),

$$\frac{D}{C}\cdot\frac{\sigma_P}{\mu_P - \frac{1}{\alpha}\overline{\mu} - \left(1-\frac{1}{\alpha}\right)r_f} = \gamma\sigma_P\,, \tag{A.10}$$

which yields the equilibrium level of μ_P and σ_P,

$$\mu_P^e = \frac{D}{\gamma C} + r_f + \frac{1}{\alpha}(\overline{\mu} - r_f)\,, \tag{A.11a}$$

$$\sigma_P^e = \sqrt{\frac{\overline{\sigma}^2}{\alpha^2} + \frac{D}{\gamma^2 C}}\,. \tag{A.11b}$$

(A.11) states that when the margin is tightened (α is increased), everything else equal, the new equilibrium portfolio would end up at the point with a lower level both of expected return and of the resturn standard deviation (i.e., southwest to the previous equilibrium portfolio).

To gauge the impact on risky market portfolio, we solve for the equilibrium risky portfolio on the original hyperpola (A.1) by utilizing the equations (A.3) to get

$$\widetilde{\mu}_P^e = \overline{\mu} + \frac{\alpha D}{\gamma C}\,, \tag{A.12a}$$

$$\widetilde{\sigma}_P^e = \sqrt{\overline{\sigma}^2 + \alpha^2\frac{D}{\gamma^2 C}}\,. \tag{A.12b}$$

Interestingly, (A.12) states that when the margin is tightened (α is increased), everything else equal, the new risky portfolio in equilibrium would end up at the point with a higher level both of expected return and of the return standard deviation (i.e., northeast to the previous risky portfolio in equilibrium).

A.3 The equations of hyperbola under differential marginability

For the ease of exposition, let us define (μ_m, σ_m) as the expected return and the return standard deviation for the marginable risky asset. For non-marginable asset, the pair of parameters is (μ_n, σ_n). We start with the hyperbola of efficient frontier comprised by these two risky assets only, characterized by a similar equation to (A.1) above (copied below),

$$\frac{\widetilde{\sigma}_P^2}{\overline{\sigma}^2} - \frac{(\widetilde{\mu}_P - \overline{\mu})^2}{\frac{D}{C} \cdot \overline{\sigma}^2} = 1\,, \quad \widetilde{\sigma}_P \geq \overline{\sigma}\,, \widetilde{\mu}_P \geq \overline{\mu}\,, \tag{A.13}$$

with its parameter values (such as C, D, $\overline{\mu}$, $\overline{\sigma}$) appropriately re-calculated based on (μ_m, σ_m) and (μ_n, σ_n).

The expected return and return variance of a portfolio consisting only of these two assets and without short-sales are:

$$\widetilde{\mu}_P = \omega\mu_m + (1-\omega)\mu_n\,, \quad 0 \leq \omega \leq 1\,, \tag{A.14a}$$

$$\widetilde{\sigma}_P^2 = \omega^2\sigma_m^2 + (1-\omega)^2\sigma_n^2 + 2\omega(1-\omega)\sigma_{mn}\,. \tag{A.14b}$$

where ω is the proportion of the portfolio held in the marginable asset, and σ_{mn} is the return covariance between these two assets. For a particular value of $\widetilde{\mu}_P$, the weight of the portfolio placed on the marginable asset is solved out from (A.14a),

$$\omega = \frac{\widetilde{\mu}_P - \mu_n}{\mu_m - \mu_n}\,. \tag{A.15}$$

Given the portfolio composition (A.15) and a margin rate α ($0 < \alpha < 1$), the investor can leverage the holdings of marginable asset up to $\frac{\omega}{\alpha}$, and the proportion $\frac{\omega}{\alpha} - \omega$ is in borrowed funds. Then, the leveraged portfolio's expected return is

$$\begin{aligned} \mu_P &= \frac{\omega}{\alpha}\mu_m + (1-\omega)\mu_n - \left(\frac{\omega}{\alpha} - \omega\right) r_f \\ &= \frac{\widetilde{\mu}_P - \mu_n}{\mu_m - \mu_n}\left[\frac{1}{\alpha}(\mu_m - r_f) - (\mu_n - r_f)\right] + \mu_n\,. \end{aligned} \tag{A.16}$$

And its return variance is

$$\begin{aligned}\sigma_P^2 &= \frac{\omega^2}{\alpha^2}\sigma_m^2 + (1-\omega)^2\sigma_n^2 + 2\frac{\omega}{\alpha}(1-\omega)\sigma_{mn} \\ &= \widetilde{\sigma}_P^2 + \omega^2\left(\frac{1}{\alpha^2}-1\right)\sigma_m^2 + 2\omega(1-\omega)\left(\frac{1}{\alpha}-1\right)\sigma_{mn}\,.\end{aligned} \tag{A.17}$$

Based upon (A.15), (A.16), and (A.17), we can express $\widetilde{\mu}_P$ and $\widetilde{\sigma}_P$ in terms of μ_P and σ_P,

$$\widetilde{\mu}_P = \mu_n + \frac{(\mu_P-\mu_n)(\mu_m-\mu_n)}{A_1}\,, \tag{A.18a}$$

$$\widetilde{\sigma}_P^2 = \sigma_P^2 - \left(\frac{\mu_P-\mu_n}{A_1}\right)^2(A_2-A_3) - \left(\frac{\mu_P-\mu_n}{A_1}\right)A_3\,, \tag{A.18b}$$

$$A_1 = \mu_m/\alpha - \mu_n - (1/\alpha-1)r_f\,, \quad A_2 = \left(\frac{1}{\alpha^2}-1\right)\sigma_m^2\,, \quad A_3 = 2\left(\frac{1}{\alpha}-1\right)\sigma_{mn}\,. \tag{A.18c}$$

and plug them back into (A.13) to get

$$\frac{\sigma_P^2 - \left(\frac{\mu_P-\mu_n}{A_1}\right)^2(A_2-A_3) - \left(\frac{\mu_P-\mu_n}{A_1}\right)A_3}{\overline{\sigma}^2} - \frac{\left(\mu_n - \overline{\mu} + \frac{(\mu_P-\mu_n)(\mu_m-\mu_n)}{A_1}\right)^2}{\frac{D}{C}\cdot\overline{\sigma}^2} = 1\,. \tag{A.19}$$

(A.19) is a very complex hyperbola equation, so is its first derivative of μ_P with respect to σ_P:

$$\frac{\partial\mu_P}{\partial\sigma_P} = \frac{\frac{D}{C}\cdot\sigma_P}{\frac{D}{C}(A_2-A_3)\frac{\mu_P-\mu_n}{A_1^2} + 2\frac{DA_3}{CA_1} + \left(\mu_n - \overline{\mu} + \frac{(\mu_P-\mu_n)(\mu_m-\mu_n)}{A_1}\right)\left(\frac{\mu_m-\mu_n}{A_1}\right)}\,. \tag{A.20}$$

Therefore, without further restrictions on these parameters, no general conclusions can be made. In particular, $\frac{\partial\mu_P}{\partial\sigma_P}$ can be positive or negative in the region of σ_P. Still, when investors prefer more returns but loathe more risks, the efficient frontier segment of (A.19) only includes the range of σ_P for which $\frac{\partial\mu_P}{\partial\sigma_P}$ is positive.